Functor of points and descent

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Schemes as functors

k = commutative ring
CAlg = category of comm. k-algebras

$$\mathrm{PreShv} := \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) = \mathrm{Fun}((\mathrm{CAlg}^{op})^{op}, \mathrm{Set})$$

<u>Def'n</u> An <u>affine scheme</u> ($/k$) is a representable functor $\mathrm{CAlg} \to \mathrm{Set}$.

$\rightsquigarrow \mathrm{AffSch} \subset \mathrm{PreShv}$ full subcat.
$\mathrm{AffSch} \cong \mathrm{CAlg}^{op}$ by Yoneda
$\quad \mathrm{Spec}\, A \leftrightarrow A$
$\Rightarrow \mathrm{PreShv} \cong \mathrm{Fun}(\mathrm{AffSch}^{op}, \mathrm{Set})$

<u>Example</u> $A = k[t_1, \dots, t_n]/(f_1, \dots, f_m)$

$$(\mathrm{Spec}\, A)(B) = \mathrm{Hom}_{\mathrm{PreShv}}(\mathrm{Spec}\, B, \mathrm{Spec}\, A)$$
$$= \mathrm{Hom}_{\mathrm{CAlg}}(A, B)$$
$$= \{(b_1, \dots, b_n) \in B^n \mid f_i(b_1, \dots, b_n) = 0 \ \forall 1 \le i \le m\}$$

<u>Example</u> V = fin.-dim. vector space (k = field)
$\rightsquigarrow \mathrm{Gr}(r, V)$ Grassmannian

$Gr(r,V)(A) = \mathrm{Hom}_{\mathrm{PreShv}}(\mathrm{Spec}\, A, Gr(r,V))$

$:= \{(E, \varphi) \mid E =$ rank r vector bundle on Spec A, $\varphi : E \to A \otimes V$ vector bundle injection$\}$

Here "rank r vector bundle" means locally free A-module of rank r, and "vector bundle injection" means that $\forall$ homomorphism $A \to K$, $K =$ field, $K \otimes_A E \to K \otimes V$ is injective. Equivalently, we require that φ is injective and that $(A \otimes V)/E$ is a flat A-module.

In particular, we have $\mathbb{P}(V) := Gr(1,V)$.

Q: How can we identify schemes among all presheaves? They are supposed to form a full subcat.

$$\mathrm{AffSch} \subset \mathrm{Sch} \subset \mathrm{PreShv}.$$

<u>Def'n</u> $X, Y \in \mathrm{AffSch}$

$\underset{\mathrm{Spec}\,A}{\overset{\shortparallel}{X}} \to \underset{\mathrm{Spec}\,B}{\overset{\shortparallel}{Y}}$ <u>closed embedding</u> $\iff B \to A$ surjective

$X, Y \in \mathrm{PreShv}$

$X \to Y$ <u>closed embedding</u> $\iff$ $\forall S \in \mathrm{AffSch}$, $S \to Y$, $S \times_Y X$ is an affine scheme and $S \times_Y X \to S$ is a closed embedding

$X, Y \in \mathrm{PreShv}$, $X \hookrightarrow Y$ monomorphism $\rightsquigarrow$ $Y \setminus X \hookrightarrow Y$

$(Y \setminus X)(S) := \{S \to Y \mid S \times_Y X = \emptyset\}$ $(\emptyset := \mathrm{Spec}\, 0)$

$X \in \mathrm{AffSch}$, $U \in \mathrm{PreShv}$

$U \hookrightarrow X$ <u>open embedding</u> $\iff$ $\exists$ closed embedding $Y \hookrightarrow X$ such that $U \cong X \setminus Y$

$X, U \in \mathrm{PreShv}$

$U \hookrightarrow X$ <u>open embedding</u> $\iff$ $\forall S \in \mathrm{AffSch}$, $S \to X$, $S \times_X U \to S$ is an open embedding

$X, U_i \in \mathrm{PreShv}$

$\{U_i \to X\}$ <u>open covering</u> $\iff$ $U_i \to X$ is an open embedding $\forall i$, and $\forall S \in \mathrm{AffSch}$, $S \to X$, $\exists i$ such that $S \times_X U_i \neq \emptyset$

$X \in \mathrm{PreShv}$ is called a <u>Zariski sheaf</u> $\iff$ $\forall S \in \mathrm{AffSch}$, $\{U_i \to S\}$ open covering with $U_i \in \mathrm{AffSch}$ $\forall i$, we have

$$X(S) \xrightarrow{\sim} \underbrace{\lim}_{\text{equalizer}} \Big(\prod_i X(U_i) \rightrightarrows \prod_{i,j} X(\overbrace{U_i \times_S U_j}^{= U_i \cap U_j}) \Big).$$

$\rightsquigarrow \mathrm{Shv}_{\mathrm{Zar}} \subset \mathrm{PreShv}$

Proposition This inclusion admits an exact left adjoint L, called (Zariski) sheafification.

Here "exact" means L preserves finite (co)limits (preservation of finite limits is the nontrivial part).

Proposition $\mathrm{AffSch} \subset \mathrm{Shv}_{\mathrm{Zar}}$

Proof Any affine scheme is a limit of copies of $\mathbb{A}^1$, and $\mathrm{Shv}_{\mathrm{Zar}}$ is stable under limits, so it suffices to show that $\mathbb{A}^1 \in \mathrm{Shv}_{\mathrm{Zar}}$. This is essentially the same calculation one does to prove that the structure presheaf is a sheaf in the locally ringed space framework. $\square$

Def'n $X \in \mathrm{Shv}_{\mathrm{Zar}}$ is called a scheme if $\exists$ open covering $\{U_i \to X\}$ with $U_i \in \mathrm{AffSch}$ $\forall i$.

Stacks

We now upgrade to functors valued in groupoids (= (1,0)-categories).

Def'n The (2,1)-category of *prestacks* is

$$\mathrm{PreStk} := \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Grpd}) \cong \mathrm{Fun}(\mathrm{AffSch}^{op}, \mathrm{Grpd})$$

Warning: since Grpd is itself a (2,1)-cat., functors into it must be defined appropriately. E.g. given $X \in \mathrm{PreStk}$ and $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$, we are supposed to have a natural isomorphism

$$\begin{tikzcd} X(S_3) \arrow[r, "X(g)"] \arrow[dr, "X(g\circ f)"'] & X(S_2) \arrow[d, "X(f)"] \arrow[dl, phantom, "\overset{\sim}{\Swarrow}"] \\ & X(S_1) \end{tikzcd}$$

satisfying coherence conditions.

Def'n $X \in \mathrm{PreStk}$ is a *Zariski*, resp. *étale*, *smooth*, *fppf stack* $\iff$ $\forall$ Zariski, resp. étale, smooth, fppf covering $\{U_i \to S\}$ in AffSch,

$$X(S) \xrightarrow{\sim} \underbrace{\lim\Big(\prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_S U_j) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k} X(U_i \times_S U_j \times_S U_k)\Big)}_{\text{"2-limit" taken in groupoids}}.$$

An object of the RHS consists of $s_i \in X(U_i)$ $\forall i$ together with isomorphisms

$$\alpha_{ij} : s_i|_{U_i \times_S U_j} \xrightarrow{\sim} s_j|_{U_i \times_S U_j} \quad \forall i,j$$

satisfying the natural cocycle condition in $X(U_i \times_S U_j \times_S U_k)$ $\forall i,j,k$.

An isomorphism $(s_i, \alpha_{ij}) \xrightarrow{\sim} (t_i, \beta_{ij})$ consists of iso.'s $\varphi_i : s_i \xrightarrow{\sim} t_i$ $\forall i$ such that

$$\begin{tikzcd} s_i|_{U_i \times_S U_j} \arrow[r, "\varphi_i"] \arrow[d, "\alpha_{ij}"'] \arrow[dr, phantom, "G"] & t_i|_{U_i \times_S U_j} \arrow[d, "\beta_{ij}"] \\ s_j|_{U_i \times_S U_j} \arrow[r, "\varphi_j"'] & t_j|_{U_i \times_S U_j} \end{tikzcd} \quad \forall i,j$$

$\rightsquigarrow$ $Stk_{fppf} \subset Stk_{sm} \underset{=}{\subset} Stk_{et} \subset Stk_{Zar} \subset PreStk$

$Shv_* = Stk_* \cap PreShv$ taken in $PreStk$

<u>Proposition</u> For $* \in \{Zar, et, fppf\}$, the inclusion

$$Stk_* \hookrightarrow PreStk$$

admits an exact left adjoint L_*.

Here "exact" and "left adjoint" should be understood in the appropriate 2-categorical sense.

<u>Example</u> G = group scheme $/k$
$(BG)_{triv} \in PreStk$ is defined by
$(BG)_{triv}(S) :=$ cat. w/ one object pt and $Aut(pt) = G(S)$
$(BG)_* := L_*(BG)_{triv} \in Stk_*$ classifies $*$-locally trivial principal G-bundles

G smooth $\Longrightarrow (BG)_{fppf} = (BG)_{ét}$
$(BGL_n)_{ét} = (BGL_n)_{Zar}$ (Hilbert 90)

For many G we have $(BG)_{ét} \neq (BG)_{Zar}$, i.e. $\exists$ étale locally trivial G-bundles which are not Zariski locally trivial.

E.g. $\overset{\gcd(n, \operatorname{char} k)=1}{}\ \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m$ is a μ_n-bundle which is étale locally trivial but not Zariski locally trivial.

$k = \mathbb{R}$, (V, q) nontrivial quadratic space $/\mathbb{R}$
$\rightsquigarrow$ $\underline{Isom}((k^n, q_{triv}), (V, q)) = O_n$-torsor which is étale locally trivial (extend scalars to $\mathbb{C}$) but not Zariski locally trivial.

<u>Theorem</u> (Grothendieck–Serre '58) $k = \bar{k}$ field. If G is semisimple and $(BG)_{ét} = (BG)_{Zar}$, then G is a product of copies of SL_n and Sp_{2n}.

In the sequel we'll simply write
$BG := (BG)_{fppf}$ $(= (BG)_{ét}$ for G smooth$)$.

Artin Stacks

Artin stacks are supposed to be "smooth locally isomorphic to an affine scheme."

G = smooth group scheme $/k$

Q: In what sense is

$$\mathrm{Spec}\, k \xrightarrow{P_{triv}} BG$$

a smooth covering?

For any $S \to BG$ classifying a G-bundle P, we have the Cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\, k \arrow[d, "P_{triv}"] \\ S \arrow[r, "P"] & BG. \end{tikzcd}$$

If G is affine, then so is P, and $P \to S$ is a smooth covering.

In general, there is some subtlety here, as the following theorem/example of Raynaud illustrates.

<u>Theorem</u> (Raynaud) $\exists$ (non-regular) commutative ring k, ~~and~~ an elliptic curve E/k, and an E-torsor P/k (in PreShv) which is not a scheme.

<u>Def'n</u> $X, Y \in \mathrm{PreStk}$
$X \to Y$ <u>schematic</u> $\Leftrightarrow \forall S \in \mathrm{AffSch}$, $S \to Y$, the prestack $S \times_Y X$ is a scheme.

So in Raynaud's example, the map $\mathrm{Spec}\, k \xrightarrow{\text{Perio}} BE$ is not schematic because P is not a scheme. In particular, it is not immediately clear what it means for this map to be a smooth covering. But fortunately, P is an algebraic space.

<u>Def'n</u> $X \in \mathrm{Shv}_{et}$ is an <u>algebraic space</u> $\Leftrightarrow$ we have

i) $\Delta : X \to X \times X$ is schematic (equivalently, $\forall S \in \mathrm{AffSch}$, $S \xrightarrow{f} X$, f is schematic)

ii) $\exists \{U_i \to X\}$, $U_i \in \mathrm{AffSch}$ $\forall i$, such that $\coprod_i U_i \to X$ (automatically schematic by (i)) is an étale covering.

$X, Y \in \mathrm{PreStk}$
$X \to Y$ <u>representable by algebraic spaces</u> $\Leftrightarrow \forall S \in \mathrm{AffSch}$, $S \to Y$, the prestack $S \times_Y X$ is an algebraic space.

If $X \xrightarrow{f} Y$ is representable by algebraic spaces, we can say what it means for f to be flat, resp. fppf, smooth, étale, etc. Namely, since these properties are supposed to be stable under base change, it suffices to consider the case where $S = Y$ is an affine scheme. Then we require that for some (equivalently, for any) étale atlas $U = \coprod_i U_i \to X$ of the alg. space X, the composite $U \to X \to Y$ is flat, resp. fppf, smooth, étale.

<u>Def'n</u> $X \in Shv_{ét}$ is an <u>Artin stack</u> $\Longleftrightarrow$ we have

i) $\Delta : X \to X \times X$ is representable by alg. spaces

ii) $\exists \{U_i \to X\}$, $U_i \in AffSch$ $\forall i$ (so $\coprod_i U_i \to X$ is representable by alg. spaces by (i)) such that $\coprod_i U_i \to X$ is a smooth covering

<u>Theorem</u> (Artin) An Artin stack is an fppf stack. In fact, X is an Artin stack $\Longleftrightarrow$ $X \in Shv_{fppf}$, (i) is satisfied, and $\exists \{U_i \to X\}$, $U_i \in AffSch$ $\forall i$, s.t. $\coprod_i U_i \to X$ is fppf.